ECE 307- Techniques for Engineering Decisions

Lecture 4. Duality Concepts in Linear Programming

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DUALITY

- **Definition: A *LP* is in *symmetric form* if all the variables are restricted to be *nonnegative* and all the constraints are inequalities of the type:**

objective type	*corresponding inequality type*
max	$\leq$
min	$\geq$

DUALITY DEFINITIONS

- We first define the *primal* and *dual* problems

$$\left.\begin{aligned} \max \quad & Z = \underline{c}^T \underline{x} \\ s.t. \quad & \\ & \underline{A}\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}\right\} (P)$$

$$\left.\begin{aligned} \min \quad & W = \underline{b}^T \underline{y} \\ s.t. \quad & \\ & \underline{A}^T \underline{y} \geq \underline{c} \\ & \underline{y} \geq \underline{0} \end{aligned}\right\} (D)$$

DUALITY DEFINITIONS

- The problems (P) and (D) are called the *symmetric dual LP* problems; we restate them as

$$
\left.
\begin{aligned}
& \max Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
& s.t. \\
& a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1 \\
& a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2 \\
& \qquad \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m \\
& x_1 \geq 0, \quad x_2 \geq 0, \quad \ldots, \quad x_n \geq 0
\end{aligned}
\right\} (P)
$$

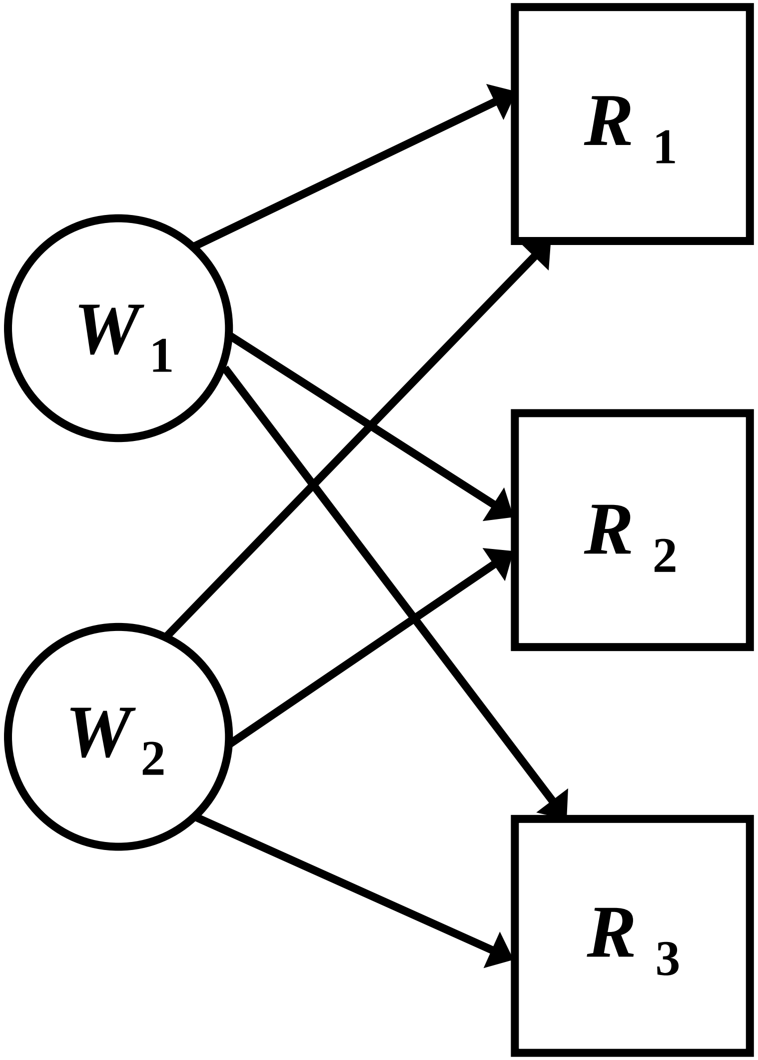
DUALITY DEFINITIONS

$$\left.\begin{aligned}
&\min W = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m \\
&s.t. \\
&a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m \geq c_1 \\
&a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m \geq c_2 \\
&\vdots \\
&a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_m \geq c_n \\
&y_1 \geq 0, \quad y_2 \geq 0, \quad \ldots, \quad y_m \geq 0
\end{aligned}\right\} (D)$$

EXAMPLE 1: MANUFACTURER TRANSPORTATION PROBLEM

shipment cost coefficients

warehouses	*retail stores*		
	R_1	R_2	R_3
W_1	2	4	3
W_2	5	3	4



EXAMPLE 1: MANUFACTURER TRANSPORTATION PROBLEM

- We are given that the *supplies* stored in warehouses W_1 and W_2 satisfy

$$\textit{supply at } W_1 \quad \leq \quad 300$$

$$\textit{supply at } W_2 \quad \leq \quad 600$$

- We are also given the *demands needed* to be met at the retail stores R_1, R_2, and R_3 :

$$\textit{demand at } R_1 \quad \geq \quad 200$$

$$\textit{demand at } R_2 \quad \geq \quad 300$$

$$\textit{demand at } R_3 \quad \geq \quad 400$$

EXAMPLE 1: MANUFACTURER TRANSPORTATION PROBLEM

- The problem is to determine the *least-cost* shipping schedule
- We define the decision variable

$$x_{ij} = \textit{quantity shipped from } W_i \textit{ to } R_j \quad i = 1,2, \;\; j = 1,2,3$$

- The shipping costs may be viewed as

$$c_{ij} = \textit{element } i, j \textit{ of the transportation cost matrix}$$

FORMULATION STATEMENT

$$\min Z = \sum_{i=1}^{2}\sum_{j=1}^{3} c_{ij}\, x_{ij} = 2x_{11} + 4x_{12} + 3x_{13} + 5x_{21} + 3x_{22} + 4x_{23}$$

s.t.

$$x_{11} + x_{12} + x_{13} \leq 300$$

$$x_{21} + x_{22} + x_{23} \leq 600$$

$$x_{11} + x_{21} \geq 200$$

$$x_{12} + x_{22} \geq 300$$

$$x_{13} + x_{23} \geq 400$$

$$x_{ij} \geq 0 \quad i = 1,2, \; j = 1,2,3$$

DUAL PROBLEM SETUP USING SYMMETRIC FORM

$$\min Z = \sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij} \, x_{ij}$$

s.t.

$$
\begin{array}{lllllllllll}
y_1 \leftrightarrow & -x_{11} & -x_{12} & -x_{13} & & & & \geq & -300 \\
y_2 \leftrightarrow & & & & -x_{21} & -x_{22} & -x_{23} & \geq & -600 \\
y_3 \leftrightarrow & x_{11} & & & +x_{21} & & & \geq & 200 \\
y_4 \leftrightarrow & & x_{12} & & & +x_{22} & & \geq & 300 \\
y_5 \leftrightarrow & & & x_{13} & & & +x_{23} & \geq & 400
\end{array}
$$

$$x_{ij} \geq 0 \quad i = 1, 2 \quad j = 1, 2, 3$$

DUAL PROBLEM SETUP

$$\max W = -300y_1 - 600y_2 + 200y_3 + 300y_4 + 400y_5$$

s.t.

$$\begin{aligned}
-y_1 \quad\quad + y_3 \quad\quad\quad\quad &\leq c_{11} = 2 \\
-y_1 \quad\quad\quad\quad + y_4 \quad\quad &\leq c_{12} = 4 \\
-y_1 \quad\quad\quad\quad\quad\quad + y_5 &\leq c_{13} = 3 \\
-y_2 + y_3 \quad\quad\quad\quad &\leq c_{21} = 5 \\
-y_2 \quad\quad + y_4 \quad\quad &\leq c_{22} = 3 \\
-y_2 \quad\quad\quad\quad + y_5 &\leq c_{23} = 4
\end{aligned}$$

$$y_i \geq 0 \quad i = 1, 2, \ldots, 5$$

THE *DUAL PROBLEM* INTERPRETATION

- The moving company proposes to the manufacturer to:

buy	all the	300	units at	W_1	at	y_1 / *unit*
buy	all the	600	units at	W_2	at	y_2 / *unit*
sell	all the	200	units at	R_1	at	y_3 / *unit*
sell	all the	300	units at	R_2	at	y_4 / *unit*
sell	all the	400	units at	R_3	at	y_5 / *unit*

- To convince the manufacturer to get the business, the mover ensures that the delivery fees cannot exceed the transportation costs the manufacturer would incur (the dual constraints)

THE *DUAL PROBLEM* INTERPRETATION

$$
\begin{array}{rcl}
-y_1 + y_3 & \le c_{11} = & 2 \\
-y_1 + y_4 & \le c_{12} = & 4 \\
-y_1 + y_5 & \le c_{13} = & 3 \\
-y_2 + y_3 & \le c_{21} = & 5 \\
-y_2 + y_4 & \le c_{22} = & 3 \\
-y_2 + y_5 & \le c_{23} = & 4
\end{array}
$$

- **The mover wishes to maximize profits, i.e.,**

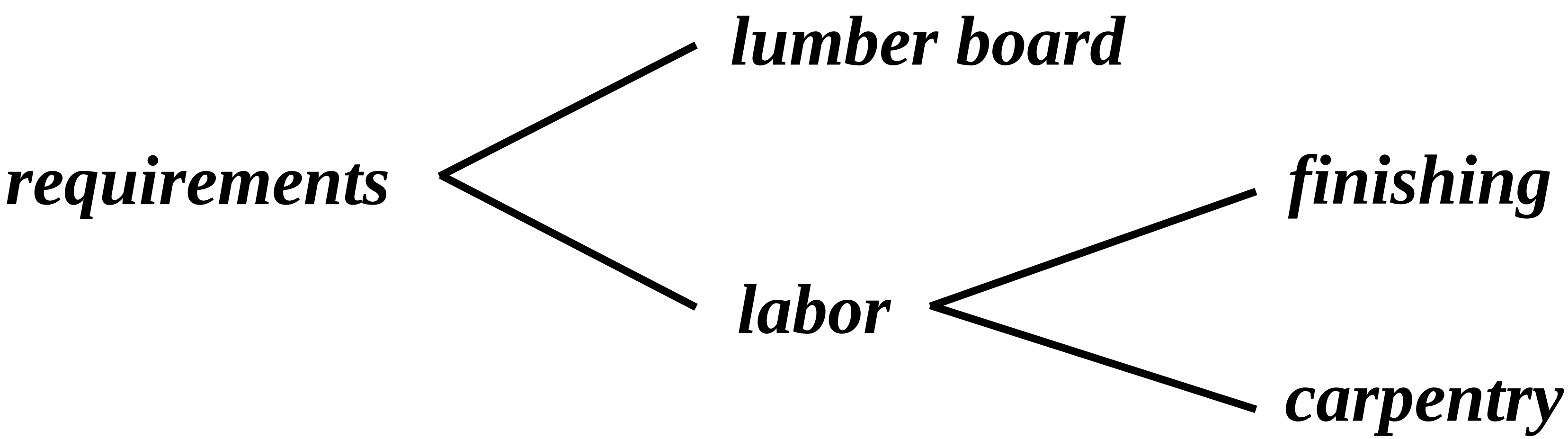
revenues – costs $\Rightarrow$ ***dual cost objective function***

$$\max W = -300\, y_1 - 600\, y_2 + 200\, y_3 + 300\, y_4 + 400\, y_5$$

EXAMPLE 2: FURNITURE PRODUCTS

- Resource requirements

item	*sales price ($)*
desks	60
tables	30
chairs	20



EXAMPLE 2: FURNITURE PRODUCTS

- The Dakota Furniture Company manufacturing:

resource	*desk*	*table*	*chair*	*available*
lumber board (*ft*)	8	6	1	48
finishing (*h*)	4	2	1.5	20
carpentry (*h*)	2	1.5	0.5	8

- We assume that the demand for desks, tables and chairs is unlimited and the available resources are already purchased
- The decision problem is to maximize *total revenues*

PRIMAL AND DUAL PROBLEM FORMULATION

- **We define decision variables**

$$x_1 = \textit{number of desks produced}$$

$$x_2 = \textit{number of tables produced}$$

$$x_3 = \textit{number of chairs produced}$$

- **The Dakota problem is**

$$max \quad Z = 60x_1 + 30x_2 + 20x_3$$

$$s.t.$$

$$y_1 \leftrightarrow 8x_1 + 6x_2 + x_3 \leq 48 \qquad lumber$$

$$y_2 \leftrightarrow 4x_1 + 2x_2 + 1.5x_3 \leq 20 \qquad finishing$$

$$y_3 \leftrightarrow 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \qquad carpentry$$

$$x_1, x_2, x_3 \geq 0$$

PRIMAL AND DUAL PROBLEM FORMULATION

- **The dual problem is**

$$\min \quad W = 48y_1 + 20y_2 + 8y_3$$

$$s.t.$$

$$8y_1 + 4y_2 + 2y_3 \geq 60 \qquad desk$$

$$6y_1 + 2y_2 + 1.5y_3 \geq 30 \qquad table$$

$$y_1 + 1.5y_2 + 0.5y_3 \geq 20 \qquad chair$$

$$y_1, y_2, y_3 \geq 0$$

PRIMAL AND DUAL PROBLEM FORMULATION

$$\max \quad Z = 60\,x_1 + 30\,x_2 + 20\,x_3$$

$$y_1 \leftrightarrow \quad 8\,x_1 + 6\,x_2 + x_3 \leq 48 \qquad \textit{lumber}$$

$$y_2 \leftrightarrow \quad 4\,x_1 + 2\,x_2 + 1.5\,x_3 \leq 20 \qquad \textit{finishing}$$

$$y_3 \leftrightarrow \quad 2\,x_1 + 1.5\,x_2 + 0.5\,x_3 \leq 8 \qquad \textit{carpentry}$$

$$x_1, x_2, x_3 \geq 0$$

$$\max \quad W = 48\,y_1 + 20\,y_2 + 8\,y_3$$

$$48\,y_1 + 20\,y_2 + 8\,y_3 \geq 60 \qquad \textit{desk}$$

$$6\,y_1 + 2\,y_2 + 1.5\,y_3 \geq 30 \qquad \textit{table}$$

$$y_1 + 1.5\,y_2 + 0.5\,y_3 \geq 20 \qquad \textit{chair}$$

$$y_1, y_2, y_3 \geq 0$$

INTERPRETATION OF THE DUAL PROBLEM

- An entrepreneur wishes to purchase all of Dakota's resources
- He needs, therefore, to determine the prices to pay for each unit of each resource

$$y_1 = \textit{price paid for 1 lumber board ft}$$

$$y_2 = \textit{price paid for 1 h of finishing labor}$$

$$y_3 = \textit{price paid for 1 h of carpentry labor}$$

- We solve the Dakota dual problem to determine y_1, y_2 and y_3

INTERPRETATION OF THE DUAL PROBLEM

- To induce Dakota to sell the raw resources, the resource prices must be set sufficiently high
- For example, the entrepreneur must offer Dakota at least $\$\,60$ for a combination of resources that consists of $8\ ft$ of lumber board, $4\ h$ of finishing and $2\ h$ of carpentry, since Dakota could use this combination to sell a desk for $\$\,60$: this requirement implies the following dual constraint:

$$8y_1 + 4y_2 + 2y_3 \geq 60$$

INTERPRETATION OF DUAL PROBLEM

- **In the same way, we obtain the two additional constraints for a table and for a chair**
- **The i^{th} primal variable is associated with the i^{th} constraint in the dual problem statement**
- **The j^{th} dual variable is associated with the j^{th} constraint in the primal problem statement**

EXAMPLE 3: DIET PROBLEM

- A new diet requires that all food eaten come from one of the four “basic food groups”:
  - chocolate cake
  - soda
  - ice cream
  - cheesecake
- The four foods available for consumption are
  - brownie
  - cola
  - chocolate ice cream
  - pineapple cheesecake

EXAMPLE 3: DIET PROBLEM

- Minimum requirements for each day are:
  - 500 *cal*
  - 6 *oz* chocolate
  - 10 *oz* sugar
  - 8 *oz* fat
- The objective is to minimize the diet costs

EXAMPLE 3: DIET PROBLEM

food	*calories*	*chocolate (oz)*	*sugar (oz)*	*fat (oz)*	*costs (cents)*
brownie	**400**	**3**	**2**	**2**	**50**
chocolate ice cream (scoop)	**200**	**2**	**2**	**4**	**20**
cola (bottle)	**150**	***0***	**4**	**1**	**30**
pineapple cheesecake (piece)	**500**	***0***	**4**	**5**	**80**

PROBLEM FORMULATION

- Objective of the problem is to minimize the total costs of the diet
- Decision variables are defined for each day's purchases

$$x_1 = \textit{number of brownies}$$

$$x_2 = \textit{number of chocolate ice cream scoops}$$

$$x_3 = \textit{number of bottles of soda}$$

$$x_4 = \textit{number of pineapple cheesecake pieces}$$

PROBLEM FORMULATION

- The problem statement is

$$\min \quad Z = 50\,x_1 + 20\,x_2 + 30\,x_3 + 80\,x_4$$

s.t.

$$
\begin{aligned}
400\,x_1 + 200\,x_2 + 150\,x_3 + 500\,x_4 &\geq 500\ cal \\
3\,x_1 + 2\,x_2 &\geq 6\ oz \\
2\,x_1 + 2\,x_2 + 4\,x_3 + 4\,x_4 &\geq 10\ oz \\
2\,x_1 + 4\,x_2 + x_3 + 5\,x_4 &\geq 8\ oz
\end{aligned}
$$

$$x_i \geq 0 \qquad i = 1,4$$

EXAMPLE 3: DIET PROBLEM

- The dual problem is

$$\max \quad W = 500\, y_1 + 6\, y_2 + 10\, y_3 + 8\, y_4$$

s.t.

$$400\, y_1 + 3\, y_2 + 2\, y_3 + 2\, y_4 \leq 50 \quad \textbf{brownie}$$

$$200\, y_1 + 2\, y_2 + 2\, y_3 + 4\, y_4 \leq 20 \quad \textbf{ice-cream}$$

$$150\, y_1 + 4\, y_3 + y_4 \leq 30 \quad \textbf{soda}$$

$$500\, y_1 + 4\, y_3 + 5\, y_4 \leq 80 \quad \textbf{cheesecake}$$

$$y_1, y_2, y_3, y_4 \geq 0$$

INTERPRETATION OF THE DUAL

- We consider a salesperson of "nutrients" who is interested in assuming that each dieter meets daily requirements by purchasing calories, sugar, fat and chocolate as "goods"
- The decision is to determine the prices charged

$$y_i \;=\; \textit{price per unit of required nutrient to sell to dieters}$$

- Objective of the salesperson is to set the prices y_i so as to maximize revenues from selling to the dieter the daily ration of required nutrients

INTERPRETATION OF DUAL

- **Now, the dieter can purchase a brownie for 50 ¢ and have 400 *cal*, 3 *oz* of chocolate, 2 *oz* of sugar and 2 *oz* of fat**
- **The sales price y_i must be set sufficiently low to entice the buyer to get the required nutrients from the brownie:**

$$400 y_1 + 3 y_2 + 2 y_3 + 2 y_4 \leq 50 \longleftarrow \text{brownie constraint}$$

- **We derive similar constraints for the ice cream, the soda and the cheesecake**

DUAL PROBLEMS

$$\left.\begin{array}{lrcl} \max & Z & = & \underline{c}^T \underline{x} \\ \text{s.t.} & & & \\ & \underline{A}\,\underline{x} & \leq & \underline{b} \\ & \underline{x} & \geq & \underline{0} \end{array}\right\} (P)$$

$$\left.\begin{array}{lrcl} \min & W & = & \underline{b}^T \underline{y} \\ \text{s.t.} & & & \\ & \underline{A}^T \underline{y} & \geq & \underline{c} \\ & \underline{y} & \geq & \underline{0} \end{array}\right\} (D)$$

WEAK DUALITY THEOREM

- For any $\underline{x}$ feasible for (P) and any $\underline{y}$ feasible for (D)

$$\underline{c}^T \underline{x} \leq \underline{b}^T \underline{y} \qquad \square$$

- Proof:

$$\underline{A}^T \underline{y} \geq \underline{c} \;\Rightarrow\; \underline{c}^T \leq \underline{y}^T \underline{A} \;\Rightarrow\; \underline{c}^T \underline{x} \leq \underline{y}^T \underline{A}\,\underline{x}$$

$$\underline{c}^T \underline{x} \leq \underline{y}^T \underline{A}\,\underline{x} \leq \underline{y}^T \underline{b} = \underline{b}^T \underline{y} \qquad \square$$

COROLLARY 1 OF THE WEAK DUALITY THEOREM

$$\underline{x} \text{ is feasible for } (P) \Rightarrow \underline{c}^T \underline{x} \leq \underline{y}^T \underline{b}$$

$$\text{for any feasible } \underline{y} \text{ for } (D)$$

$$\underline{c}^T \underline{x} \leq \underline{y}^{*T} \underline{b} = \min W$$

$$\text{for any feasible } \underline{x} \text{ for } (P),$$

$$\underline{c}^T \underline{x} \leq \min W$$

COROLLARY 2 OF THE WEAK DUALITY THEOREM

$$\underline{y} \text{ is feasible for } (D) \Rightarrow \underline{c}^T \underline{x} \leq \underline{y}^T \underline{b}$$

$$\text{for every feasible } \underline{x} \text{ for } (P)$$

$$\max Z = \max \underline{c}^T \underline{x} = \underline{c}^T \underline{x}^* \leq \underline{y}^T \underline{b}$$

$$\text{for any feasible } \underline{y} \text{ of } (D),$$

$$\underline{y}^T \underline{b} \geq \max Z$$

COROLLARIES 3 AND 4 OF THE WEAK DUALITY THEOREM

If (P) is feasible and $max\ Z$ is unbounded, i.e.,

$$Z \rightarrow +\infty ,$$

then, (D) has no feasible solution.

If (D) is feasible and $min\ Z$ is unbounded, i.e.,

$$Z \rightarrow -\infty ,$$

then, (P) is infeasible.

DUALITY THEOREM APPLICATION

- Consider the maximization problem

$$\left.\begin{aligned}
&\max Z = x_1 + 2x_2 + 3x_3 + 4x_4 = \underbrace{\begin{bmatrix}1, 2, 3, 4\end{bmatrix}}_{\underline{c}^T} \underline{x} \\
&s.t. \\
&\underbrace{\begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 3 & 2 \end{bmatrix}}_{\underline{A}} \underline{x} \leq \underbrace{\begin{bmatrix} 20 \\ 20 \end{bmatrix}}_{\underline{b}} \\
&\underline{x} \geq \underline{0}
\end{aligned}\right\} (P)$$

DUALITY THEOREM APPLICATION

- The corresponding dual is given by

$$\left.\begin{array}{ll} \min & W = \underline{b}^T \underline{y} \\ s.t. & \\ & \underline{A}^T \underline{y} \geq \underline{c} \\ & \underline{y} \geq \underline{0} \end{array}\right\} (D)$$

- With the appropriate substitutions, we obtain

DUALITY THEOREM APPLICATION

$$\min \quad W = 20 y_1 + 20 y_2$$

s.t.

$$y_1 + 2 y_2 \geq 1$$

$$2 y_1 + y_2 \geq 2$$

$$2 y_1 + 3 y_2 \geq 3$$

$$3 y_1 + 2 y_2 \geq 4$$

$$y_1 \geq 0, \quad y_2 \geq 0$$

GENERALIZED FORM OF THE DUAL

- Consider the primal decision

$$x_i = 1, \quad i = 1, 2, 3, 4 \; ;$$

decision is feasible for (P) with

$$Z = \underline{c}^T \underline{x} = 10$$

- The dual decision

$$y_i = 1, \quad i = 1, 2$$

is feasible for (D) with

$$W = \underline{b}^T \underline{y} = 40$$

DUALITY THEOREM APPLICATION

- **Clearly,**

$$Z\left(x_1, x_2, x_3, x_4\right) = 10 \leq 40 = W\left(y_1, y_2\right)$$

and so clearly, the feasible decision for (*P*) and (*D*)

satisfy the *Weak Duality Theorem*

- **Moreover, we have**

$$corollary\ 1 \Rightarrow 10 \leq min\, W = W\left(y^*_1, y^*_2\right)$$

$$corollary\ 2 \Rightarrow max\ Z = Z\left(x^*_1, x^*_2, x^*_3, x^*_4\right) \leq \underline{b}^T \underline{y} = 40$$

COROLLARIES 5 AND 6

(P) is feasible and (D) is infeasible, then,

(P) is unbounded □

(D) is feasible and (P) is infeasible, then,

(D) is unbounded □

EXAMPLE

- Consider the primal dual problems:

$$\left.\begin{aligned} &\max Z = x_1 + x_2 \\ &s.t. \\ &-x_1 + x_2 + x_3 \leq 2 \\ &-2x_1 + x_2 - x_3 \leq 1 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}\right\} (P) \qquad \left.\begin{aligned} &\min W = 2y_1 + y_2 \\ &s.t. \\ &-y_1 - 2y_2 \geq 1 \\ &y_1 + y_2 \geq 1 \\ &y_1 - y_2 \geq 0 \\ &y_1, y_2 \geq 0 \end{aligned}\right\} (D)$$

- Now

$$\underline{x} = \underline{0} \text{ is feasible for } (P)$$

EXAMPLE

$$\underline{x} = \underline{0} \text{ is feasible for } (P)$$

but

$$-y_1 - 2y_2 \geq 1$$

is impossible for (D) since it is inconsistent with

$$y_1, y_2 \geq 0$$

- Since (D) is infeasible, it follows from Corollary 5 that $Z \to \infty$
- You are able to show this result by solving (P) using the simplex scheme

OPTIMALITY CRITERION THEOREM

- We consider the primal-dual problems (P) and (D) with

$$\left.\begin{array}{l} \underline{x}^{0} \text{ is feasible for } (P) \\ \underline{y}^{0} \text{ is feasible for } (D) \\ \underline{c}^{T}\underline{x}^{0} = \underline{b}^{T}\underline{y}^{0} \end{array}\right\} \Rightarrow \begin{array}{l} \underline{x}^{0} \text{ is optimal for } (P) \\ \text{and} \\ \underline{y}^{0} \text{ is optimal for } (D) \end{array}$$

- We next provide the proof:

$$\begin{array}{l} \underline{x}^{0} \text{ is feasible for } (P) \\ \underline{y}^{0} \text{ is feasible for } (D) \end{array} \underset{Theorem}{\overset{Weak\ \ Duality}{\Rightarrow}} \underline{c}^{T}\underline{x}^{0} \leq \underline{b}^{T}\underline{y}^{0}$$

OPTIMALITY CRITERION THEOREM

but we are given that

$$\underline{c}^{T}\underline{x}^{0} = \underline{b}^{T}\underline{y}^{0}$$

and so it follows that $\forall$ *feasible* $\underline{x}$ *with* $\underline{y}^{0}$ *feasible*

$$\underline{c}^{T}\underline{x} \leq \underline{b}^{T}\underline{y}^{0} = \underline{c}^{T}\underline{x}^{0}$$

and so $\underline{x}^{0}$ is *optimal* ;

similarly, $\forall$ *feasible* $\underline{y}$ *with* $\underline{x}^{0}$ *feasible*

$$\underline{b}^{T}\underline{y} \geq \underline{c}^{T}\underline{x}^{0} = \underline{b}^{T}\underline{y}^{0}$$

and so it follows that $\underline{y}^{0}$ is *optimal*

MAIN DUALITY THEOREM

(P) is feasible and (D) is feasible; then,

$\exists \, \underline{x}^{*}$ *feasible for* (P) *which is optimal and*

$\exists \, \underline{y}^{*}$ *feasible for* (D) *which is optimal such that*

$$\underline{c}^{T} \underline{x}^{*} = \underline{b}^{T} \underline{y}^{*}$$

COMPLEMENTARY SLACKNESS CONDITIONS

- $\underline{x}^*$ and $\underline{y}^*$ are optimal for (P) and (D), respectively, if and only if

$$0 = \left(\underline{y}^{*T}\underline{A} - \underline{c}^{T}\right)\underline{x}^{*} + \underline{y}^{*T}\left(\underline{b} - \underline{A}\,\underline{x}^{*}\right)$$

$$= \underline{y}^{*T}\underline{b} - \underline{c}^{T}\underline{x}^{*}$$

- We prove this equivalence result by defining the slack variables $\underline{u} \in \mathbb{R}^{m}$ and $\underline{v} \in \mathbb{R}^{n}$ such that $\underline{x}$ and $\underline{y}$ are feasible; at the optimum,

$$\underline{A}\,\underline{x}^{*} + \underline{u}^{*} = \underline{b} \qquad \underline{x}^{*},\ \underline{u}^{*} \geq \underline{0}$$

$$\underline{A}^{T}\underline{y}^{*} - \underline{v}^{*} = \underline{c} \qquad \underline{y}^{*},\ \underline{v}^{*} \geq \underline{0}$$

COMPLEMENTARY SLACKNESS CONDITIONS

where the optimal values of the slack variables

$\underline{u}^*$ and $\underline{v}^*$ depend on the optimal values

$\underline{x}^*$ and $\underline{y}^*$

- Now,

$$\underline{y}^{*T}\underline{A}\underline{x}^* + \underline{y}^{*T}\underline{u}^* = \underline{y}^{*T}\underline{b} = \underline{b}^T\underline{y}^*$$

$$\underbrace{\underline{x}^{*T}\underline{A}^T\underline{y}^*}_{\underline{y}^{*T}\underline{A}\underline{x}^*} - \underline{x}^{*T}\underline{v}^* = \underline{x}^{*T}\underline{c} = \underline{c}^T\underline{x}^*$$

COMPLEMENTARY SLACKNESS CONDITIONS

- This implies that

$$\underline{y}^{*T}\underline{u}^{*} + \underline{v}^{*T}\underline{x}^{*} = \underline{b}^{T}\underline{y}^{*} - \underline{c}^{T}\underline{x}^{*}$$

- We need to prove optimality which is true if and only if

$$\underline{y}^{*T}\underline{u}^{*} + \underline{v}^{*T}\underline{x}^{*} = 0$$

COMPLEMENTARY SLACKNESS CONDITIONS

- However,

$$\underline{x}^*, \underline{y}^* \text{ are optimal}$$

$$\xRightarrow[\text{Duality Theorem}]{\text{Main}}$$

$$\underline{c}^T \underline{x}^* = \underline{b}^T \underline{y}^* \Rightarrow \underline{y}^{*T} \underline{u}^* + \underline{v}^{*T} \underline{x}^* = 0$$

- Also,

$$\underline{y}^{*T} \underline{u}^* + \underline{v}^{*T} \underline{x}^* = 0 \Rightarrow \underline{b}^T \underline{y}^* = \underline{c}^T \underline{x}^*$$

$$\xRightarrow[\text{Criterion Theorem}]{\text{Optimality}}$$

$$\underline{x}^* \text{ is optimal for } (P) \text{ and } \underline{y}^* \text{ is optimal for } (D)$$

COMPLEMENTARY SLACKNESS CONDITIONS

- Note that

$$\underline{x}^*, \underline{y}^*, \underline{u}^*, \underline{v}^* > 0 \Rightarrow \textit{component - wise each element} \geq 0$$

$$\underline{y}^{*T}\underline{u}^* + \underline{v}^*\underline{x}^* = 0 \Rightarrow y_i^* u_i^* = 0 \;\; \forall i = 1, \ldots, m$$

$$\textit{and} \;\; v_j^* x_j^* = 0 \;\; \forall j = 1, \ldots, n$$

- At the optimum,

$$y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij} x_j^* \right) = 0 \quad i = 1, \ldots, m$$

and

$$x_j^* \left(\sum_{i=1}^{m} a_{ji} y_i^* - c_j \right) = 0 \quad j = 1, \ldots, n$$

COMPLEMENTARY SLACKNESS CONDITIONS

- Hence, for $i = 1, 2, \ldots, m$

$$y_i^* > 0 \Rightarrow b_i = \sum_{j=1}^{n} a_{ij} x_j^*$$

and

$$b_i - \sum_{j=1}^{m} a_{ij} x_i^* > 0 \Rightarrow y_i^* = 0$$

- Similarly for $j = 1, 2, \ldots, n$

$$x_i^* > 0 \Rightarrow \sum_{i=1}^{m} a_{ji} y_i^* = c_j$$

and

$$\sum_{i=1}^{m} a_{ji} y_i^* - c_j > 0 \Rightarrow x_j^* = 0$$

EXAMPLE

$$
\left.\begin{aligned}
&\max \quad Z = x_1 + 2x_2 + 3x_3 + 4x_4 \\
&s.t. \\
&\qquad x_1 + 2x_2 + 2x_3 + 3x_4 \leq 20 \\
&\qquad 2x_1 + x_2 + 3x_3 + 2x_4 \leq 20 \\
&\qquad x_i \geq 0 \quad i = 1, \ldots, 4
\end{aligned}\right\} (P)
$$

EXAMPLE

$$
\left.\begin{aligned}
\min \quad & W = 20y_1 + 20y_2 \\
s.t. \quad & \\
& y_1 + 2y_2 \geq 1 \\
& 2y_1 + y_2 \geq 2 \\
& 2y_1 + 3y_2 \geq 3 \\
& 3y_1 + 2y_2 \geq 4 \\
& y_1, y_2 \geq 0
\end{aligned}\right\} (D)
$$

EXAMPLE

$\underline{x}^*, \underline{y}^*$ *optimal* $\Rightarrow$

$$y_1^* \left(20 - x_1^* - 2x_2^* - 2x_3^* - 3x_4^* \right) = 0$$

$$y_2^* \left(20 - 2x_1^* - x_2^* - 3x_3^* - 2x_4^* \right) = 0$$

$\underline{y}^* = \begin{bmatrix} 1.2 \\ 0.2 \end{bmatrix}$ **is given as an optimal solution with**

$$\min W = 28$$

EXAMPLE

$$x_1^* + 2x_2^* + 2x_3^* + 3x_4^* = 20$$

$$2x_1^* + x_2^* + 3x_3^* + 2x_4^* = 20$$

$$y_1^* + 2y_2^* = 1.2 + 0.4 > 1 \Rightarrow x_1^* = 0$$

$$2y_1^* + y_2^* = 2.4 + 0.2 > 2 \Rightarrow x_2^* = 0$$

$$2y_1^* + 3y_2^* = 2.4 + 0.6 = 3$$

$$3y_1^* + 2y_2^* = 3.6 + 0.4 = 4$$

so that

$$\left.\begin{array}{ll} 2x_3^* + 3x_4^* = 20 & \Rightarrow x_3^* = 4 \\ 3x_3^* + 2x_4^* = 20 & \Rightarrow x_4^* = 4 \end{array}\right\}$$

COMPLEMENTARY SLACKNESS CONDITION APPLICATIONS

- **Key uses of *c.s. conditions* are**
  - **finding optimal (*P*) solution given optimal (*D*) solution and vice versa**
  - **verification of optimality of solution (whether a feasible solution is optimal)**
- **We can start with a feasible solution and attempt to construct an optimal dual solution; if we succeed, then the feasible primal solution is *optimal***

DUALITY

$$(P)\quad \begin{aligned} \max \quad & Z = \underline{c}^T \underline{x} \\ \text{s.t.} \quad & \\ & \underline{A}\,\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

$$(D)\quad \begin{aligned} \min \quad & W = \underline{b}^T \underline{y} \\ \text{s.t.} \quad & \\ & \underline{A}^T \underline{y} \geq \underline{c} \\ & \underline{y} \geq \underline{0} \end{aligned}$$

DUALITY

- Suppose the primal problem is minimization, then,

$$\min \quad Z = \underline{c}^T \underline{x} \qquad (P)$$

$$s.t.$$

$$\underline{A}\,\underline{x} \geq \underline{b}$$

$$\underline{x} \geq \underline{0}$$

$$\max \quad W = \underline{b}^T \underline{y} \qquad (D)$$

$$s.t.$$

$$\underline{A}^T \underline{y} \leq \underline{c}$$

$$\underline{y} \geq \underline{0}$$

INTERPRETATION

- The economic interpretation is

$$Z^{*} = \max Z = \underline{c}^{T}\underline{x}^{*} = \underline{b}^{T}\underline{y}^{*} = W^{*} = \min W$$

$$\left.\begin{array}{l} b_i - \text{constrained resource quantities,} \\ y_i^{*} - \text{optimal dual variables} \end{array}\right\} \quad i = 1, 2, \ldots, m$$

- Suppose, we change

$$b_i \rightarrow b_i + \Delta b_i \Rightarrow \Delta Z = y_i^{*} \Delta b_i$$

- In words, the optimal dual variable for each primal constraint gives the net change in the optimal value of the objective function Z for a one unit change in the constraint on resources

INTERPRETATION

- Economists refer to the dual variable as the *shadow price* on the constraint resource
- The *shadow price* determines the value/worth of having an additional quantity of a resource
- In the previous example, the optimal dual variables indicate that the worth of another unit of resource 1 is 1.2 while that of another unit of resource 2 is 0.2

GENERALIZED FORM OF THE DUAL

- We start out with

$$\left.\begin{array}{llll} max & Z & = & \underline{c}^{T}\underline{x} \\ s.t. & & & \\ & \underline{A}\,\underline{x} & = & \underline{b} \\ & \underline{x} & \geq & \underline{0} \end{array}\right\} (P)$$

GENERALIZED FORM OF THE DUAL

- To find (D), we first put (P) in symmetric form

$$\begin{array}{rcrcr} \underline{y}_{+} & \leftrightarrow & \underline{A}\,\underline{x} & \leq & \underline{b} \\ \underline{y}_{-} & \leftrightarrow & -\underline{A}\,\underline{x} & \leq & -\underline{b} \\ & & \underline{x} & \geq & \underline{0} \end{array} \qquad \begin{bmatrix} \underline{A} \\ \\ -\underline{A} \end{bmatrix} \underline{x} \leq \begin{bmatrix} \underline{b} \\ \\ -\underline{b} \end{bmatrix}$$

symmetric form

GENERALIZED FORM OF THE DUAL

- Let

$$\underline{y} = \underline{y}_{+} - \underline{y}_{-}$$

- We rewrite the problem as

$$min\ W = \underline{b}^{T} \underline{y}$$

$$s.t.$$

$$\underline{A}^{T} \underline{y} \geq \underline{c}$$

$$\underline{y} \text{ is unsigned}$$

- The *c.s.* conditions apply

$$\underline{x}^{*T} \left(\underline{A}^{T} \underline{y}^{*} - \underline{c} \right) = \underline{0}$$

EXAMPLE 5: THE PRIMAL

$$\max Z = x_1 - x_2 + x_3 - x_4$$

$$s.t.$$

$$\begin{array}{lclr}
y_1 & \leftrightarrow & x_1 + x_2 + x_3 + x_4 = & 8 \\
y_2 & \leftrightarrow & x_1 \leq & 8 \\
y_3 & \leftrightarrow & x_2 \leq & 4 \\
y_4 & \leftrightarrow & -x_2 \leq & 4 \\
y_5 & \leftrightarrow & x_3 \leq & 4 \\
y_6 & \leftrightarrow & -x_3 \leq & 2 \\
y_7 & \leftrightarrow & x_4 \leq & 10 \\
 & & x_1, x_4 \geq & 0 \\
 & & x_2, x_3 & \text{unsigned}
\end{array} \qquad (P)$$

EXAMPLE 5: THE DUAL

$$\min W = 8y_1 + 8y_2 + 4y_3 + 4y_4 + 4y_5 + 2y_6 + 10y_7\pi$$

s.t.

$$x_1 \leftrightarrow \quad y_1 + y_2 \qquad\qquad \geq 1$$

$$x_2 \leftrightarrow \quad y_1 \qquad + y_3 - y_4 \qquad = -1 \qquad (D)$$

$$x_3 \leftrightarrow \quad y_1 \qquad\qquad + y_5 - y_6 \qquad = 1$$

$$x_4 \leftrightarrow \qquad\qquad\qquad\qquad + y_7 \geq 1$$

$$y_2, \ldots\ldots, y_7 \geq 0$$

$$y_1 \text{ unsigned}$$

EXAMPLE 5: *c.s. conditions*

- We are given that

$$\underline{x}^{*} = \begin{bmatrix} 8 \\ -4 \\ 4 \\ 0 \end{bmatrix}$$

is optimal for (P)

- Then the *c.s. conditions* obtain

$$x_1^{*}\left(y_1^{*} + y_2^{*} - 1 \right) = 0$$

EXAMPLE 5: *c.s. conditions*

so that

$$x_1^* = 8 > 0 \Rightarrow y_1^* + y_2^* = 1$$

- The other *c.s. conditions* obtain

$$y_i^* \left(\sum_{j=1}^{4} a_{ij} x_j^* - b_i \right) = 0$$

- Now, $x_4^* = 0$ implies $x_4^* - 10 < 0$ and so

$$y_7^* = 0$$

EXAMPLE 5: *c.s. conditions*

- Also, $x_3^* = 4$ implies

$$y_6^* = 0$$

- Similarly, the *c.s. conditions*

$$x_j^* \left(\sum_{i=1}^{7} a_{ji}\, y_i^* - c_j \right) = 0$$

have implications on the y_i^* variable

EXAMPLE 5: *c.s. conditions*

- Since $x_2^* = -4$, then we have

$$y_3^* = 0$$

- Now, with $y_7^* = 0$ we have

$$y_1^* > -1$$

- Since, $W = \underline{b}^T \underline{y}$ we have

$$y_2^* = 1 - y_1^*$$

EXAMPLE 5

- Suspose

$$y_1^* = 1$$

and so,

$$y_2^* = 0$$

- Furthermore,

$$y_1^* + y_3^* - y_4^* = 1 - y_4^* = -1$$

implies

$$y_4^* = 2$$

EXAMPLE 5

- Also

$$y_1^* + y_5^* - y_6^* = 1$$

implies

$$1 + y_5^* = 1$$

and so

$$y_5^* = 0$$

EXAMPLE 5

- Therefore

$$W\left(\underline{y}^{*}\right) = (8)(1)+(8)(0)+(4)(0)+(4)(2)+$$

$$(4)(0)+(2)(0)+(10)(0)$$

$$= 16$$

and so

$$W^{*} = 16 = Z^{*} \Leftrightarrow \text{ optimality of } (P) \text{ and } (D)$$

PRIMAL – DUAL TABLE

primal (maximize)	*dual* (minimize)
$\underline{A}$ (coefficient matrix)	$\underline{A}^T$ (transpose of the coefficient matrix)
$\underline{b}$ (right-hand side vector)	$\underline{b}$ (cost vector)
$\underline{c}$ (price vector)	$\underline{c}$ (right hand side vector)
i^{th} constraint is $=$ type	the dual variable y_i is unrestricted in sign
i^{th} constraint is $\leq$ type	the dual variable $y_i \geq 0$
i^{th} constraint is $\geq$ type	the dual variable $y_i \leq 0$
x_j is unrestricted	j^{th} dual constraint is $=$ type
$x_j \geq 0$	j^{th} dual constraint is $\geq$ type
$x_j \leq 0$	j^{th} dual constraint is $\leq$ type